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CLOSURE OPERATORS I

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Closure operators in an (E, M)-category X are introduced as concrete endofunctors of the comma category whose objects are the elements of M. Various kinds of closure operators are studied. There is a Galois equivalence between the conglomerate of idempotent and weakly hereditary closure operators of X and the conglomerate of subclasses of M which are part of a factorization system. There is a one-to-one correspondence between the class of regular closure operators and the class of strongly epireflective subcategories of X. Every closure operator admits an idempotent hull and a weakly hereditary core.

Various examples of additive closure operators in **Top** are given. For abelian categories standard closure operators are considered. It is shown that there is a one-to-one correspondence between the class of standard closure operators and the class of preradicals. Idempotent, weakly hereditary, standard closure operators correspond to idempotent radicals (= torsion theories).

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closure operator	θ-closure	(strongly) epireflective subcateogry
idempotent hull	A-regular morphism	weakly hereditary core
sequential closure	(pre)-radical	factorization system

Introduction

Closure operators have always been one of the main concepts in topology. For example, Herrlich [21] characterized coreflections in the category **Top** of topological spaces by means of Kuratowski closure operators finer than the ordinary closure. Hong [25] and Salbany [35] used closure operators to produce epireflective subcategories of **Top**. Nakagawa [32] gave a Galois equivalence between certain factorization systems in **Top** and additive, weakly hereditary and idempotent closure operators. This idea was developed later by Lord [31] for certain concrete categories (see also [23]). Closure operators have been used also to generalize the well-known fact that a space X is Hausdorff iff the diagonal Δ_X is closed in $X \times X$. These results will be referred to as 'diagonal theorems' (cf. [9, 10, 12, 16, 18, 19, 20, 23, 24, 36,

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42]. On the other hand, the study of epimorphisms and the co-well-poweredness of subcategories of **Top** in [8, 10, 12, 13, 14, 15, 16, 17, 18, 19, 36, 37, 42] is heavily based on closure operators.

We propose here an abstract notion of closure operator which covers all previous kinds of closure operators and provides a unified approach to many different, apparently distant aspects of mathematics. Due to the lack of space we shall give here only the basic notions and properties of closure operators as well as many examples. The second part of this paper, containing various applications (epimorphisms, co-well-poweredness, absolutely closed objects (see [15]), connectedness-disconnectedness etc.) will appear elsewhere.

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1. Preliminaries

Throughout the paper we consider a category $\boldsymbol{X}$ and a fixed class $\boldsymbol{M}$ of morphisms in $\boldsymbol{X}$ which contains all isomorphisms of $\boldsymbol{X}$. It is assumed that

(i) $\boldsymbol{M}$ is closed under composition, and that

(ii) $\boldsymbol{X}$ is $\boldsymbol{M}$-complete, that is: pullbacks of $\boldsymbol{M}$-morphisms exist and belong to $\boldsymbol{M}$, and multiple pullbacks of (possibly large) families of $\boldsymbol{M}$-morphisms with common codomain exist and belong to $\boldsymbol{M}$.

We list some consequences of these hypotheses (cf. [40] in the dual situation):

(iii) every morphism in $\boldsymbol{M}$ is a monomorphism in $\boldsymbol{X}$;

(iv) if $nm \in \boldsymbol{M}$ and $n \in \boldsymbol{M}$, then also $m \in \boldsymbol{M}$;

(v) for each object X in $\boldsymbol{X}$, the comma category $\boldsymbol{M}_X$ of $\boldsymbol{M}$-morphisms with codomain X is a (possibly large) complete preordered set. We shall use the usual lattice-theoretical notations ($\leqslant$, $\wedge$, $\vee$, $\bigwedge$, $\bigvee$) in $\boldsymbol{M}_X$;

(vi) for every morphism $f: X \to Y$ in $\boldsymbol{X}$, there is an 'inverse image' functor $f^{-1}(-): \boldsymbol{M}_Y \to \boldsymbol{M}_X$ give by pullback. $f^{-1}(-)$ has a left adjoint $f(-)$, the 'image' under f;

(vii) there is a (uniquely determined) class $\boldsymbol{E}$ of morphisms in $\boldsymbol{X}$ such that $(\boldsymbol{E}, \boldsymbol{M})$ is a factorization system of $\boldsymbol{X}$, that is: every morphism has an $(\boldsymbol{E}, \boldsymbol{M})$-factorization, and the $(\boldsymbol{E}, \boldsymbol{M})$-diagonalization property holds: whenever $he = mg$ with $e \in \boldsymbol{E}$ and $m \in \boldsymbol{M}$, then there is a unique t with $te = g$ and $mt = h$. In the terminology of (vi) if $f = me$ with $e \in \boldsymbol{E}$ and $m \in \boldsymbol{M}$, then $m \cong f(1_X)$;

(viii) property (vii) holds more generally for arbitrary sinks, in which case $\boldsymbol{E}$ has to be extended to a conglomerate of sinks. The conjunction of properties (i) and (ii) is equivalent to (viii).

The class $\boldsymbol{M}$ will be considered as the class of objects of a comma category which is denoted again by $\boldsymbol{M}$. Its morphisms $(f, g): m \to n$ are commutative squares

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "m"'] & \bullet \arrow[d, "n"] \\ \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

The categories $\boldsymbol{M}_X$ are the (nonfull) subcategories of $\boldsymbol{M}$ for which $g = 1_X$; they are the fibres of the 'codomain functor' $U: \boldsymbol{M} \to \boldsymbol{X}$ $(U(f, g) = g)$. U is faithful by (iii), so $\boldsymbol{M}$ is concrete over $\boldsymbol{X}$. However, by (viii) one has the much stronger statement that U is a topological functor (cf. [39, 22]). In particular U has a full and faithful left adjoint D which assigns to each $\boldsymbol{X}$-object X its least $\boldsymbol{M}$-subobject, and a full and faithful right adjoint I.

2. Closure operators

A *closure operator* of the category $\boldsymbol{X}$ with respect to the class of subobjects $\boldsymbol{M}$ is a concrete functor

$$C: \ \boldsymbol{M} \to \boldsymbol{M},$$

so $UC = U$, such that there is a natural transformation $\gamma: \mathrm{Id}_{\boldsymbol{M}} \to C$ with $U\gamma = 1_U$. Since U is faithful, γ is uniquely determined and a pointwise bimorphism. The pair (C, γ) is therefore a prereflection in the sense of [41] and, a fortiori, a well-pointed endofunctor in the sense of [30]. Recall from [41] that (C, γ) is a reflection if and only if $\gamma C (= C\gamma)$ is an isomorphism, and that in this case

$$\mathrm{Fix}(C, \gamma) = \{m \in \boldsymbol{M} : \gamma(m) \text{ is an isomorphism}\}$$

is (the class of objects of) a full bireflective subcateogry of $\boldsymbol{M}$.

Restriction of C gives, for each object X, operators

$$C_X: \ \boldsymbol{M}_X \to \boldsymbol{M}_X;$$

we write $([m]:[M] \to X) := C(m: M \to X)$ for $m \in \boldsymbol{M}$, $[m]$ is called *C-closure* of m. Whenever needed sub- or superscripts are added: $[m]_X^C = [m]_X = [m]^C = [m]$. If we have $m \leq n$ in $\boldsymbol{M}_X$, so there is a unique k with $nk = m$, then we put

$$m_n := k.$$

With this notation $\gamma(m)$ is given by the square

$$\begin{tikzcd} M \arrow[r, "m_{[m]}"] \arrow[d, "m"'] & {[M]} \arrow[d, "{[m]}"] \\ X \arrow[r, "1_X"] & X \end{tikzcd} \qquad (*)$$

In particular, one has

(1) $\quad m \leq [m].$

By functoriality of C, for every $f:X\to Y$ and $m\in \boldsymbol{M}_X$, $n\in \boldsymbol{M}_Y$, one also has

(2) $\quad m\leqslant f^{-1}(n)\Rightarrow[m]\leqslant f^{-1}([n])$.

Of course, (2) is equivalent to (2a) and (2b):

(2a) $\quad m\leqslant m'\Rightarrow[m]\leqslant[m']$,

(2b) $\quad [f^{-1}(n)]\leqslant f^{-1}([n])$.

By adjointness, (2) is also equivalent to

(3) $\quad f(m)\leqslant n\Rightarrow f([m])\leqslant[n]$

which is equivalent to (3a) = (2a) and

(3b) $\quad f([m])\leqslant[f(m)]$.

Notice that the closure operators of $\boldsymbol{X}$ with respect to $\boldsymbol{M}$ are in one-to-one correspondence to families of operators $\{[\ \ ]_X:\boldsymbol{M}_X\to \boldsymbol{M}_X\}_{X\in \mathrm{Ob}\boldsymbol{X}}$ which satisfy conditions (1) and (2) above.

The conglomerate of all closure operators of $\boldsymbol{X}$ with respect to $\boldsymbol{M}$ can be endowed with the 'pointwise' preorder defined by $C\leqslant D$ if and only if $C(m)\leqslant D(m)$ for all $m\in \boldsymbol{M}$, and will be denoted by $\mathrm{CL}(\boldsymbol{X},\boldsymbol{M})$. If (C,γ) and (D,δ) are closure operators and $C\leqslant D$ then $\eta(m):=([m]^C)_{[m]}D$, $m\in \boldsymbol{M}$, define a natural transformation $\eta:C\to D$ such that $n\gamma=\delta$. Arbitrary infima (and suprema) exist in $\mathrm{CL}(\boldsymbol{X},\boldsymbol{M})$ since they exist in each $\boldsymbol{M}_X$ and are preserved by $f^{-1}(\text{-})$. In particular, there is a last closure operator, namely IU (see Section 1), and a least one, namely Id_M; the former is called *trivial*, the latter *discrete*.

3. (C-dense, C-closed)-factorizations

Diagram (∗) of Section 2 gives, for every $m\in \boldsymbol{M}$, a factorization $m=[m]\cdot m_{[m]}$. We shall give minimal conditions such that this is a factorization into a 'dense' and a 'closed' factor, according to the following definitions. For $C\in\mathrm{CL}(\boldsymbol{X},\boldsymbol{M})$, any $m\in M$ with $m\cong[m]$ is called *C-closed*, and a morphism $d:X\to Y$ in $\boldsymbol{X}$ is *C-dense* if $[d(1_X)]\cong 1_Y$.

C is called *idempotent* if, for every $m\in \boldsymbol{M}$, $[m]$ is C-closed, and it is called *weakly hereditary* if, for every $m\in \boldsymbol{M}$, $m_{[m]}$ is C-dense.

Remark. C is idempotent if and only if the prereflection (C,γ) is a reflection (see Section 2). $\mathrm{Fix}(C,\gamma)$ contains exactly the C-closed elements of $\boldsymbol{M}$. In the following, we shall denote it by $\boldsymbol{M}^C$, and the class of C-dense morphisms in $\boldsymbol{X}$ will be denoted by $\boldsymbol{E}^C$. Obviously $\boldsymbol{M}^C\subseteq \boldsymbol{M}$ and $E\subseteq \boldsymbol{E}^C$.

3.1. Proposition. *X has the (E^C, M^C)-diagonalization property.*

Proof. If $mg = hd$ with $m \in M^C$ and $d \in E^C$, one considers an (E, M)-factorization $d = ne$, so $n = d(1)$. By assumption, $m_{[m]}$ and $[n]$ are isomorphisms. The (E, M)-diagonalization property gives a morphism t with $te = g$ and $mt = hn$. From the latter equation, by functoriality of C, one obtains a t' with $[m]t' = h[n]$. Since m is monic, the morphism $s = (m_{[m]})^{-1}t'[n]^{-1}$ is the only morphism that satisfies $sd = g$ and $ms = h$.

3.2. Prosposition. *For an idempotent closure operator C, the following assertions are equivalent:*

(i) *C is weakly hereditary,*

(ii) *M^C is closed under composition in X,*

(iii) *X has (E^C, M^C)-factorizations.*

Proof. (i)$\Rightarrow$(iii): let $f = em$ be an (E, M)-factorization of f. Then $f = [m](m_{[m]}e)$ is an (E^C, M^C)-factorization of f.

(iii)$\Rightarrow$(ii) follows from Proposition 3.1.

(ii)$\Rightarrow$(i): for $m \in M$ and $k = m_{[m]}$ one has that $[m][k]$ is C-closed. So from $m \leq [m][k]$ we can deduce $[m] \leq [m][k]$, thus $[k]$ is an isomorphism.

3.3. Corollary. *For an idempotent and weakly hereditary closure operator C one has that E^C and M^C are closed under composition in X, M^C is stable under pullbacks and the formation of limits in X; E^C has the dual properties.*

For an arbitrary factorization system (E', M') of X with $M' \subseteq M$ one can define $C(m) = m'$, where $m = m'e'$ is an (E', M')-factorization of $m \in M$. Trivially, C is an idempotent and weakly hereditary closure operator with $M^C = M'$. Together with Proposition 3.2 one therefore obtains:

3.4. Theorem. *There is a Galois equivalence between the conglomerate of subclasses of M which are part of a factorization system, and the conglomerate of idempotent and weakly hereditary closure operators of X with respect to M. In both conglomerates, arbitrary infima and suprema exist.*

Proof. Infima of subclasses M' of M which are part of a factorization system, can be formed by intersection: these are those M' which are closed under composition and stable under pullbacks and multiple pullbacks (see Section 1). □

Results similar to Theorem 3.4 can be found in [32, 31, 23].

In addition to the properties (1)–(3b) of Section 2, for every closure operator one also has

(4) $\quad m \leq n \Rightarrow n[m_n] \leq n \wedge [m],$

(5) $\quad [m] \vee [n] \leq [m \vee n]$

for all m, $n \in \boldsymbol{M}_X$, $X \in \mathrm{Ob}\,\boldsymbol{X}$. In fact, C transforms

$$\begin{tikzcd} \bullet \arrow[r, "1"] \arrow[d, "m_n"'] & \bullet \arrow[d, "m"] \\ \bullet \arrow[r, "n"] & \bullet \end{tikzcd} \quad \text{into} \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "{[m_n]}"'] & \bullet \arrow[d, "{[m]}"] \\ \bullet \arrow[r, "n"] & \bullet \end{tikzcd}$$

so $n[m_n] \leqslant [m]$, and trivially $n[m_n] \leqslant n$. (5) is trivial.

A closure operator C is called *hereditary* if $n[m_n] \cong n \wedge [m]$ holds for all m, $n \in \boldsymbol{M}_X$ with $m \leqslant n$, and *additive* if $[m] \vee [n] \cong [m \vee n]$ holds for all m, $n \in \boldsymbol{M}_X$, $X \in \mathrm{Ob}\,\boldsymbol{X}$.

3.5. Proposition. *Let C be idempotent. Then:*

(1) *C is weakly hereditary if and only if $n[m_n] \cong n \wedge [m]$ holds whenever $m \leqslant n$ and n is C-closed.*

(2) *C is hereditary if and only if it is weakly hereditary and satisfies $n[m_n] \cong n \wedge [m]$ whenever $m \leqslant n$ and n is C-dense.*

(3) *C is additive if and only if the class $\boldsymbol{M}_X^C$ of C-closed elements in $\boldsymbol{M}_X$ is closed under binary suprema for every $X \in \mathrm{Ob}\,\boldsymbol{X}$.*

The easy proofs are left to the reader.

4. Idempotent hull and weakly hereditary core

If $\{C_i\}_{i \in I}$ is a family of closure operators of $\boldsymbol{X}$ with respect to $\boldsymbol{M}$, then one easily shows

(I) $\bigwedge_{i \in I} C_i$ is idempotent if each C_i is,

(II) $\bigvee_{i \in I} C_i$ is weakly hereditary if each C_i is.

One therefore has the following proposition.

4.1. Proposition. (1) *The conglomerate* ID-CL($\boldsymbol{X}, \boldsymbol{M}$) *of idempotent closure operators is reflective in* CL($\boldsymbol{X}, \boldsymbol{M}$). *The reflector sends C to its idempotent hull*

$$\hat{C} := \bigwedge\{D \colon C \leqslant D,\ D \text{ idempotent}\}.$$

(2) *The conglomerate* WH-CL($\boldsymbol{X}, \boldsymbol{M}$) *of weakly hereditary closure operators is coreflective in* CL($\boldsymbol{X}, \boldsymbol{M}$). *The coreflector sends C to its weakly hereditary core*

$$\check{C} := \bigvee\{D \colon D \leqslant C,\ D \text{ weakly hereditary}\}.$$

We shall give a more concrete description of $\hat{C}$ and $\check{C}$ in case $\boldsymbol{X}$ is $\boldsymbol{M}$-well-powered, that is if each $\boldsymbol{M}_X$ has a small skeleton. For every $C \in \mathrm{CL}(\boldsymbol{X}, \boldsymbol{M})$ one defines an ascending chain of closure operators C^α by

$$C^1 = C, \qquad C^{\alpha+1} = C \circ C^\alpha, \qquad C^\beta = \bigvee_{\alpha<\beta} C^\alpha$$

for every (small) ordinal α and every limit ordinal β, and for $\beta = \infty$ (with $\infty > \alpha$ for all small α). If $\boldsymbol{X}$ is $\boldsymbol{M}$-well-powered, for every $X \in \mathrm{Ob}\,\boldsymbol{X}$, there is an α such that $C^{\alpha+1}(m) = C^\alpha(m)$ for all $m \in \boldsymbol{M}_X$; the least such α is called the *order of* $\boldsymbol{X}$ with respect to C and it is denoted by $O_C(X)$. So C^∞ is idempotent, and for every idempotent $D \geq C$ one has $C^\infty \leq D$. If there is an ordinal α such that $C^\alpha = C^\infty$, then the least such ordinal is called the *order of* C and it is denoted by $O(C)$; otherwise C is called *unbounded.*

One also has a descending chain C_α of closure operators, defined by

$$C_1 = C, \quad C_\beta = \bigwedge_{\beta<\alpha} C_\alpha \quad \text{and} \quad C_{\alpha+1}(m) = C_\alpha(m)C(m_\alpha)$$

with $m_\alpha := m_{C_\alpha(m)}$, for all $m \in \boldsymbol{M}$ and for α and β as above. If $\boldsymbol{X}$ is $\boldsymbol{M}$-well-powered, for every $X \in \mathrm{Ob}\,\boldsymbol{X}$, there is an α such that $C(m_\alpha)$ is an isomorphism, so m_α is C-dense for all $m \in \boldsymbol{M}_X$. It follows that C_∞ is weakly hereditary, and that $C_\infty \geq D$ for every weakly hereditary $D \leq C$. This shows part (1) of the following

4.2. Theorem. *Let* $\boldsymbol{X}$ *be* $\boldsymbol{M}$*-well-powered and* $C \in \mathrm{CL}(\boldsymbol{X}, \boldsymbol{M})$. *Then*

(1) $\hat{C} \cong C^\infty$ *and* $\check{C} \cong C_\infty$,

(2) *if* C *is weakly hereditary, also* $\hat{C}$ *is,*

(3) *if* C *is idempotent, also* $\check{C}$ *is.*

Proof. (2): because of part (1) and property (II) it suffices to show that C^2 is weakly hereditary if C is. So let $m \in \boldsymbol{M}$, and put $k = m_{[m]}$ and $j = [m]_{[[m]]}$. Then $jk = m_{[[m]]}$, and C transforms the commutative square

$$\begin{tikzcd} \bullet \arrow[r, "1"] \arrow[d, "k"'] & \bullet \arrow[d, "jk"] \\ \bullet \arrow[r, "j"] & \bullet \end{tikzcd} \quad \text{into} \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "{[k]}"'] & \bullet \arrow[d, "{[jk]}"] \\ \bullet \arrow[r, "j"] & \bullet \end{tikzcd}$$

with $[k]$ an isomorphism. So $j \leq [jk]$, hence $[j] \leq [[jk]]$ with $[j]$ an isomorphism. Therefore, also $[[jk]]$ is an isomorphism, so $m_{[[m]]}$ is C^2-dense.

(3): it suffices to show that C_2 is idempotent if C is. For $m \in \boldsymbol{M}$, as above we put $k = m_{[m]}$, so $C_2(m) = [m][k]$. Furthermore, let $h = ([m][k])_{[[m][k]]}$, so $C_2(C_2(m))) = [[m[k]][h]$. By assumption, there are isomorphisms $j: [m] \to [[m]]$ and $i: [k] \to [[k]]$. Since $[m][k] \leq [m]$ and, therefore, $[[m[k]] \leq [[m]]$, there is also a morphism r

rendering the diagram

$$\begin{tikzcd}
\bullet \arrow[rr, "1"] \arrow[d, "h"'] & & \bullet \arrow[d, "{[k]}"] \\
\bullet \arrow[r, "r"] \arrow[dr, "{[[m][k]]}"'] & \bullet \arrow[r, "j^{-1}"] \arrow[d, "{[[m]]}"] & \bullet \arrow[dl, "{[m]}"] \\
& \bullet &
\end{tikzcd}$$

commutative. C transforms its upper part into a commutative square

$$\begin{tikzcd}
\bullet \arrow[r, "s"] \arrow[d, "{[h]}"'] & \bullet \arrow[d, "{[[k]]}"] \\
\bullet \arrow[r, "j^{-1}r"] & \bullet
\end{tikzcd}$$

From

$$[[m][k]][h]=[[m]]r[h]=[m]j^{-1}r[h]=[m][[k]]s=[m][k]i^{-1}s=[[m][k]]hi^{-1}s$$

one concludes $[h]\leqslant h$, thus $h\cong[h]$. Consequently, $C_2(C_2(m))\cong C_2(m)$. $\square$

4.3. Corollary. *For every $C\in \mathrm{CL}(\boldsymbol{X},\boldsymbol{M})$,*

$$\hat{\check{C}}\cong\check{C} \quad \textit{and} \quad \check{\hat{C}}\cong\hat{C}.$$

Remarks. (1) The construction of C^{∞} is presented, in a more general context, in [41, 30].

(2) For $\boldsymbol{X}$ $\boldsymbol{M}$-well-powered one has, for every $m\in\boldsymbol{M}$,

$$\hat{C}(m)=\bigwedge\{n\in\boldsymbol{M}_X^C\colon m\leqslant n\}$$

since the right-hand side is easily seen to coincide with $C^{\infty}(m)$ (up to isomorphism). Therefore $\hat{C}_X$ is the reflector of $\boldsymbol{M}_X$ into $\boldsymbol{M}_X^C$.

(3) $\hat{C}$ is additive if C is and $\check{C}$ is hereditary if C is.

5. Regular closure operators

In what follows $\boldsymbol{X}$ is a category with finite products and $\boldsymbol{M}$ contains the class of regular monomorphisms of $\boldsymbol{X}$.

For a subcategory $\boldsymbol{A}$ of $\boldsymbol{X}$, a morphism $r: X \to Y$ is an *A-regular* monomorphism if it is the equalizer of two morphisms $f, g: Y \to A$, with $A \in \boldsymbol{A}$. For $m: M \to X$ in $\boldsymbol{M}$ set

$$[m]^{A} = \bigwedge \{r\colon r \geqslant m \text{ and } r \text{ is } \boldsymbol{A}\text{-regular}\}$$

Then $C_{\boldsymbol{A}}(m) = [m]^{A}$ defines a concrete functor $C_{\boldsymbol{A}}: \boldsymbol{M} \to \boldsymbol{M}$ which is a closure operator of $\boldsymbol{X}$. These closure operators will be called *regular* and $C_{\boldsymbol{A}}(m)$ will be called the A-closure of m.

Regular closure operators where introduced in [35] for $\boldsymbol{X} = \mathbf{Top}$, in [16] for $\boldsymbol{X}$ a topological category, and recently in [9] in a more general context.

For a closure operator C of $\boldsymbol{X}$, set

$$\mathscr{D}(C) = \{X\colon \Delta_X \text{ is } C\text{-closed}\}, \qquad \mathscr{C}(C) = \{X\colon \Delta_X \text{ is } C\text{-dense}\}.$$

If $C = C_{\boldsymbol{A}}$ we write $\mathscr{D}(\boldsymbol{A})$ and $\mathscr{C}(\boldsymbol{A})$ instead of $\mathscr{D}(C_{\boldsymbol{A}})$ and $\mathscr{C}(C_{\boldsymbol{A}})$. For a subcategory $\boldsymbol{A}$ of $\boldsymbol{X}$ set

$$S(\boldsymbol{A}) = \{X\colon \text{there is a mono-source } \{f_i: X \to A_i\}_{i \in I} \text{ with } A_i \in \boldsymbol{A}\}.$$

If $\boldsymbol{A}$ is reflective with reflector r and reflexion $\rho: 1_X \to r$, then $S(\boldsymbol{A})$ consists of all objects X of $\boldsymbol{X}$ such that ρ_X is a monomorphism. The following condition will be considered below:

(+) for each $X, Y \in \operatorname{Ob} \boldsymbol{X}$, the canonical morphism $k: r(X \times Y) \to rX \times rY$ is a monomorphism.

5.1. Proposition. (1) $C_{\boldsymbol{A}} = C_{S(\boldsymbol{A})}$;

(2) *$\mathscr{D}(\boldsymbol{A})$ is closed under mono-sources*;

(3) *$\mathscr{C}(\boldsymbol{A})$ is closed under E-morphisms* (*see Section* 1), *whenever* ***E*** *is closed under finite products*;

(4) *$\mathscr{D}(\boldsymbol{A}) \cap \mathscr{C}(\boldsymbol{A})$ is the subcategory of quasiterminal objects*;

(5) *If* ***A*** *is reflective in* ***X*** *and satisfies* (+), *then $\mathscr{D}(\boldsymbol{A}) = S(\boldsymbol{A})$*;

(6) *If* ***X*** *has* (*strong epi, mono-source*)*-factorizations and every epireflective subcategory of* ***X*** *satisfies* (+), *then the assignments $\boldsymbol{A} \to C_{\boldsymbol{A}}$ $C \to \mathscr{D}(C)$ determine a Galois correspondence which is a Galois equivalence between the class of strongly epireflective subcategories of* ***X*** *and the class of regular closure operators of* ***X***.

Proof. The easy proofs of (1)–(4) are left to the reader. (5) is Theorem 1.1 of [20]. (6): in the case $\boldsymbol{X}$ has (strong epi, mono-source)-factorizations, $\mathscr{D}(\boldsymbol{A})$ is strongly epireflective in $\boldsymbol{X}$, according to (2). Thus (6) follows from (1), (2) and (5). □

Remarks. (1) Under suitable conditions, the pair $(\mathscr{D}(C), \mathscr{C}(C))$ gives a sort of 'disconnectedness' related to C (see [4] for disconnectedness in **Top**). In fact $\mathscr{C}(C)$ is the connectedness generated by $\mathscr{D}(C)$.

(2) For a subcategory $\boldsymbol{A}$ of $\boldsymbol{X}$ the epimorphisms in $\boldsymbol{A}$ are exactly the $C_{\boldsymbol{A}}$-dense morphisms. This simple characterization of epimorphisms in subcategories $\boldsymbol{A}$ of $\boldsymbol{X}$

was the starting point for the study of regular closure operators in [17] and [12] (see also the references in [18]).

(3) The $\boldsymbol{A}$-closure in concrete categories is determined by its restriction on $\boldsymbol{A}$. Precisely, if $\boldsymbol{A}$ is epireflective in $\boldsymbol{X}$ then, for each $m \in \boldsymbol{M}$, $[m]^{\boldsymbol{A}} = \rho X^{-1}([\rho(m)]^{\boldsymbol{A}})$, where ρ is the $\boldsymbol{A}$-reflection (cf. [13, 16]).

6. Examples

6.1. Let $\boldsymbol{X}$ be an abelian category and let $\boldsymbol{M}$ be the class of monomorphisms in $\boldsymbol{X}$. Then every closure operator C of $\boldsymbol{X}$ gives rise to a preradical r_C in $\boldsymbol{X}$ defined, for each $X \in \operatorname{Ob} \boldsymbol{X}$, by $r_C(X) = [o]^C$, $o : O \to X$ (for pre-radicals in abelian categories see [11]). r_C has the following relation with C. For $X \in \operatorname{Ob} \boldsymbol{X}$ and $m \in \boldsymbol{M}_X$ let $\phi : X \to X/M$ be the cokernel of $m : M \to X$. Then

$$[m]^C \leqslant \phi^{-1}(r_C(X/_M)). \qquad (**)$$

An example in which (**) is not an isomorphism is given below. Closure operators for which (**) is an isomorphism for each $X \in \operatorname{Ob} \boldsymbol{X}$ will be called *standard.* Conversely, for every preradical r in $\boldsymbol{X}$, the closure operator C_r defined, for every $m \in \boldsymbol{M}_X$, by $[m]^{C_r} = \phi^{-1}(r(X/M))$ is standard. Thus there is a one-to-one correspondence between the class of preradicals in $\boldsymbol{X}$ and the class of standard closure operators of $\boldsymbol{X}$.

For every preradical r in $\boldsymbol{X}$ one defines two series of preradicals setting, for every $X \in \operatorname{Ob} \boldsymbol{X}$, (here $\boldsymbol{X}$ is supposed to be well-powered)

$$r^1(X) = r(X), \qquad r^{\alpha+1}(X) = r(X/_{r^\alpha(X)}), \qquad r^\beta(X) = \bigvee_{\alpha<\beta} r^\alpha(X);$$

$$r_1(X) = r(X), \qquad r_{\sigma+1}(X) = r(r_\alpha(X)), \qquad r_\beta(X) = \bigwedge_{\alpha<\beta} r_\alpha(X).$$

for every (small) ordinal α and every limit ordinal β, and for $\beta = \infty$. Clearly $C_{r^\alpha} = C_r^\alpha$, $C_{r_\alpha} = (C_r)_\alpha$, $r_C\alpha = (r_C)^\alpha$, $r_{C_\alpha} = (r_C)_\alpha$, and r^∞, r_∞ are respectively, the radical hull and the weakly hereditary core of r. Then $C_{r^\infty} = (C_r)^\infty$ and $C_{r_\infty} = (C_r)_\infty$; in particular C_r is idempotent if and only if r is a radical and it is weakly hereditary if and only if r is idempotent. $\{r^\alpha(X)\}$ is known as the Loewy series of X and its length, which coincides with $O_{C_r}(X)$ defined in Section 4, as the Loewy's length of X.

C_r is hereditary if and only if r is hereditary, that is: $r(M) = r(X) \wedge m$, for every $m : M \to X$ in $\boldsymbol{M}$. This shows in particular, that non hereditary closure operators which are weakly hereditary exist in profusion (see [6]).

The idempotent weakly hereditary standard closure operators correspond to idempotent radicals (=torsion theories) in $\boldsymbol{C}$. It follows from Theorem 3.4 that every torsion theory provides a factorization system in $\boldsymbol{X}$.

There exist idempotent and hereditary nonadditive closure operators. Take $\boldsymbol{X}=$ **Ab**, the category of abelian groups and the radical t given, for every abelian group X, by its torsion part $t(X)$. If X is the subgroup of $\mathbb{Q}^2$ generated by $\mathbb{Z}^2$ and the diagonal $\Delta_{\mathbb{Q}}$, then $m:Z\times O\to X$ and $n:O\times Z\to X$ are C_t-closed while $m\vee n$ is C_t-dense.

For $X\in\boldsymbol{X}$ and $m\in\boldsymbol{M}_X$, m is C_r-dense (C_r-closed) if and only if $r(X/M)=1_{X/M}$ ($r(X/M)=O$). Thus $S(C_r)=\{X:r(X)=o\}$ and $T(C_r)=\{X:\ r(X)=1_X\}$.

Finally we give examples of nonstandard closure operators. Let r be a radical in $\boldsymbol{X}$ and let s be a preradical with $s\leqslant r$. Then, for each monomorphism $m:M\to X$ in $\boldsymbol{X}$, define $C_{r,s}$ by, $[m]^{C_{r,s}}\geqslant n$ and $[m]^{C_{r,s}}/_n=s(X/M+r(X))$, where n is the embedding of $M+r(X)$ in X. For $r=s$, $F_{r,s}=F_r$ is standard and $F_{r,s}$ is standard if and only if $F_{r,s}=F_r$. For $\boldsymbol{X}=$ **Ab**, $r=t$ as above, and s given by the p-torsion part, p any prime number, $F_{r,s}$ is not standard.

6.2. Let $\boldsymbol{X}$ be a category of universal algebras (in particular of semigroups or rings) and let $\boldsymbol{M}$ be the class of monomorphisms in $\boldsymbol{X}$. For $m\in\boldsymbol{M}_X$, $[m]^{C_X}$ is known as the dominion of m in X (see [27]). Dominions can be described to some extent and in particular it can be shown that if for $m:M\to X$ in $\boldsymbol{M}$, M infinite, the cardinality of M is greater or equal to the number of algebraic operations, then $\text{card}[M]^{C_X}=\text{card}\,M$. In particular $\boldsymbol{X}$ is co-well-powered (cf. [27]). When $\boldsymbol{X}$ is the category of all semigroups, then dominions are described in a very elegant way by the 'zigzag' Theorem 2.3 in [27]. For epimorphisms and dominions in the category of finite-dimensional algebras over a field see [28] and the references therein.

6.3. In what follows let $\boldsymbol{X}=$ **Top** and let $\boldsymbol{M}$ be the class of embeddings in **Top**. Let K denote the closure operator of **Top** given by the Kuratowski operator of the ordinary closure. K will play a very important role. Notice that there exist non-additive regular closure operators in **Top** (*cf.* [8]) and, there exist also regular closure operators which are not comparable with K (cf. [19]).

Since in **Top** there is a one-to-one correspondence between the set $\boldsymbol{M}_X$ of the embeddings in X and the set of subsets of X, then any closure operator of **Top** is determined, for each $m:M\to X$ in $\boldsymbol{M}_X$, by the underlying set of $[M]$. Thus we will write $M\subset X$ for m or for the underlying set of M. Notice also that, any regular closure operator $C_{\boldsymbol{A}}$ of **Top** is determined by its restriction to $\boldsymbol{A}$ (Section 5, Remark (3)). By the fact that every nontrivial coreflective subcategory of **Top** is bicoreflective, it follows that $|\emptyset|^C_X=\emptyset$ for every weakly hereditary non trivial closure operator C and for every $X\in$ **Top**. So the idempotent hull of every additive and weakly hereditary closure operator is a weakly hereditary Kuratowski operator.

6.3.A. Additive closure operators coarser than K.

(1) For $\boldsymbol{A}=$ **Haus** (Hausdorff spaces), **SHaus** (strongly Hausdorff spaces (cf. [32])), $\mathbf{Top}_3$ (regular Hausdorff spaces), **Tych** (completely regular Hausdorff spaces), 0-**dim** (0-dimensional Hausdorff spaces), the $\boldsymbol{A}$-closure coincides with K on $\boldsymbol{A}$-spaces (cf. 12]).

(2) For a topological space X and $M \subset X$ set

$$Z(M) = \bigcap \{N \subset X\colon M \subset N \text{ and } N \text{ is a zero-set}\}.$$

Z is an additive, idempotent, nonweakly hereditary closure operator and $\mathcal{D}(Z) = \mathbf{FT}_2$, the category of functionally Hausdorff spaces (the continuous real-valued functions separate the points).

(3) For a topological space X and $M \subset X$, set

$$C(M) = \bigcap \{N \subset X\colon M \subset N \text{ and } N \text{ is clopen}\}.$$

C is additive, idempotent and $\mathcal{D}(C) = S(0\text{-}\mathbf{dim})$ (spaces in which the quasicomponents are single points). This is the coarsest nontrivial regular closure operator.

(4) For an epireflective subcategory $\boldsymbol{A}$ of **Top** and $X \in S(\boldsymbol{A})$, the $S(\boldsymbol{A})$-closure on X coincides with the $\boldsymbol{A}$-closure on the $\boldsymbol{A}$-reflexion of X. In particular, since $\mathbf{FT}_2 = S(\mathbf{Tych})$, the Z-closure in $\mathbf{FT}_2$-spaces is the K-closure in the **Tych**-reflexions.

(5) Let α be an ordinal number. For a topological space X and $M \subset X$, set

$$\mathrm{cl}_{\theta^\alpha} M = \{x \in X\colon \text{for each chain of open neighbourhoods } \{U_\beta\colon \beta < \alpha\} \text{ of } x \\ \text{with } \bar{U}_{\beta+1} \subset U_\beta \text{ for } \beta + 1 < \alpha,\ \bar{U}_0 \cap M \neq \emptyset\}.$$

Obviously, for every α, $\mathrm{cl}_{\theta^\alpha}$ is an additive closure operator coarser than K (for $\alpha = 1$ it is the well-known θ-closure defined in [43]).
For $\alpha \geq \omega$ $\mathcal{D}(\mathrm{cl}_{\theta^\alpha})$ coincides with $S(\alpha)$ defined in [34]. For finite $\alpha = k$, $\mathcal{D}(\mathrm{cl}_{\theta^k}) = S(2k)$ and, for every $n > 0$ (X, τ) is a $S(n)$-space if and only if the topology in X determined by the cl_{θ^n}-closed sets of X is T_0 (cf. [10]). $S(1) = \mathbf{Haus}$ and $S(2)$ is the category of Urysohn spaces. For $\alpha \geq \omega$ the $S(\alpha)$-closure coincides on $S(\alpha)$-spaces with $\mathrm{cl}_{\theta^\alpha}$, for $\alpha = n < \omega$, the $S(n)$-closure coincides on $S(n)$-spaces with cl_{θ^k}, where k is the greatest integer less or equal to $\frac{1}{2}n$ (cf. [12] for $n = 1$, [36] for $n = 2$, [15] for $n < \omega$, [10] for arbitrary α).

The closure operator cl_θ is not bounded. In [37] this fact was used to show that $S(2)$ is not co-well-powered.

(6) The following closure operators were defined inductively in [38]: define first h_0 as the discrete operator and, for $n \geq 0$, X a topological space and $M \subset X$, set $h_{n+1}(M) = \{x \in X\colon$ for each open neighbourhood U of $h_n(U) \bigcap h_n(M) \neq \emptyset\}$; then the (additive) operator h'_n is defined by

$$h'_n(M) = \{x \in X\colon \text{for each open neighbourhood } U \text{ of } x,\ h_n(U) \bigcap M \neq \emptyset\}.$$

$\mathcal{D}(h'_n)$ coincides with the S_n-spaces defined in [1], the S_n-closure coincides with h'_n on S_n-spaces, and, for $n > 1$, S_n is not co-well-powered (cf. [38]). By Proposition 5.1.(2) S_n is strongly epireflective in **Top**.

6.3.B. Additive closure operators finer than K.

(1) The well-known b-closure C_b (cf. [5]) is the finest nondiscrete regular closure operator, $\mathcal{D}(C_b) = \{T_0\text{-spaces}\}$ and $\mathcal{C}(C_b) = \{\text{indiscrete spaces}\}$. Moreover C_b is a hereditary Kuratowski operator.

(2) For a class $\boldsymbol{P}$ of nonempty topological spaces define the following closure operators ($X \in \mathbf{Top}$ and $M \subset X$):

$$\mathrm{cl}_{\boldsymbol{P}}(M) = \bigcup \{\overline{f(P) \cap M} : P \in \boldsymbol{P}, f : P \to X\},$$

$$\mathrm{Cl}_{\boldsymbol{P}}(M) = \bigcup \{\overline{\overline{f(P)} \cap M} \cap f(P) : P \in \boldsymbol{P}, f : P \to X\},$$

$$d_{\boldsymbol{P}}(M) = \bigcup \{\overline{f(P) \cap M} \cap f(P) : P \in \boldsymbol{P}, f : P \to X\},$$

$$\mathrm{cl}^{\boldsymbol{P}}(M) = \bigcup \{f(\overline{f^{-1}(M)}) : P \in \boldsymbol{P}, f : P \to X\},$$

$$\mathrm{Cl}^{\boldsymbol{P}}(M) = p(\mathrm{cl}^{\boldsymbol{P}}(k_0(X)) \cap \mathrm{cl}^{\boldsymbol{P}}(k_1(X))),$$

where k_0 and k_1 are the natural embeddings of X in the adjunction space $X \sqcup_M X$ and $p : X \sqcup_M X \to X$ is the natural projection. In fact $\mathrm{cl}^{\boldsymbol{P}}$ is the closure operator corresponding to the coreflective subcategory of **Top** generated by $\boldsymbol{P}$ in the sense of [21].

In general these operators are additive, and

$$\mathrm{cl}_{\boldsymbol{P}} \geq \mathrm{Cl}_{\boldsymbol{P}} \geq d_{\boldsymbol{P}} \geq \mathrm{cl}^{\boldsymbol{P}} \leq \mathrm{Cl}^{\boldsymbol{P}}$$

holds. If $\boldsymbol{P}$ is closed under closed subspaces (arbitrary subspaces), then, except for the last one, they are all weakly hereditary (hereditary).

Consider the following condition on $\boldsymbol{P}$ (cf. [24]):

(C) for every $P \in \boldsymbol{P}$, there exists a nonempty space F such that $P \cup F \in \boldsymbol{P}$. If $\boldsymbol{P}$ satisfies (C), then $\mathrm{cl}_{\boldsymbol{P}} = \mathrm{Cl}_{\boldsymbol{P}}$. If $\boldsymbol{P}$ is closed under continuous images then $d_{\boldsymbol{P}} = \mathrm{cl}^{\boldsymbol{P}}$. If $\boldsymbol{P}$ is closed under finite coproducts then $\mathrm{Cl}^{\boldsymbol{P}} \leq \mathrm{Cl}_{\boldsymbol{P}}$ and if $\boldsymbol{P}$ is closed both under continuous images and finite coproducts, then $\mathrm{cl}_{\boldsymbol{P}} = \mathrm{Cl}_{\boldsymbol{P}} = \mathrm{Cl}^{\boldsymbol{P}}$ and $d_{\boldsymbol{P}} = \mathrm{cl}^{\boldsymbol{P}}$.

$\mathscr{D}(\mathrm{cl}^{\boldsymbol{P}})$ is the category (denoted by $\boldsymbol{P}_3$ in [24] and [14]) of all spaces X for which the diagonal in $X \times X$ is closed in the coreflection generated by $\boldsymbol{P}$. If $\boldsymbol{P}$ is closed under closed subspaces and satisfies (C), then $C_{\boldsymbol{P}_3}$ coincides with $\widehat{\mathrm{Cl}^{\boldsymbol{P}}}$ on the subcategory $\boldsymbol{P}_3$. In the particular case $\boldsymbol{P} = \{\mathbb{N}_\infty\}$, where $\mathbb{N}_\infty$ is the one-point compactification of $\mathbb{N}$, $\mathrm{cl}^{\boldsymbol{P}} = d_{\boldsymbol{P}} = \sigma$ is the sequential closure operator which is additive, hereditary and $\mathrm{O}(\sigma) \leq \omega_1$. $\mathscr{D}(\sigma)$ is the category of all spaces in which the convergent sequences have unique limit points. It is easy to see that, for $X \in \mathbf{Top}$ and $M \subset X$, $\mathrm{Cl}^{\{N_\infty\}}(M) = \{x \in X : x = \lim x_n \text{ and } x \in \overline{\{x_{n_k}\}} \cap M \text{ for each subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\}$. This closure operator is not weakly hereditary. Analogous results are valid if one replaces $\{N_\infty\}$ by any class of right filtered sets which is closed under taking cofinal sets (cf. [26]).

The operator $\mathrm{cl}_{\boldsymbol{P}}$ is unbounded in the case $\boldsymbol{P}$ is the class of all compact spaces (cf. [29, 19]). On the other hand, if the density of every space in $\boldsymbol{P}$ is less or equal to some cardinal $\mathfrak{m}$, then $|\mathrm{O}(\mathrm{cl}_{\boldsymbol{P}})| \leq (2^{2^{\mathfrak{m}}})^+$ (cf. [14]).

(3) For a class $\boldsymbol{P}$ as above let $\boldsymbol{P}_2$ denote the class of all spaces X such that, for every $P \in \boldsymbol{P}$ and every continuous map $f : P \to X$, the subspace $f(P)$ of X is Hausdorff (cf. [24]; in [14] this category is denoted by $\mathbf{Haus}(\boldsymbol{P})$). $\boldsymbol{P}_2$ is strongly epireflective

in **Top** and satisfies **Haus** $\subset \boldsymbol{P}_2 \subset \{T_1\text{-spaces}\}$, unless $\boldsymbol{P}$ consists of indiscrete spaces only. If $\boldsymbol{P}$ satisfies (C), then $\boldsymbol{P}_2 \subset \boldsymbol{P}_3$ and $C_{\boldsymbol{P}_2} = \mathrm{cl}_{\boldsymbol{P}}$ on the subcategory $\boldsymbol{P}_2$. Many examples of concrete $\boldsymbol{P}$ can be found in [14], [19] and [24].

(4) Let $\mathscr{m}$ be a cardinal number; a subset U of a topological space X is a $G_{\mathscr{m}}$-set if U is the intersection of fewer than $\mathscr{m}$ open sets. The $\mathscr{m}$-closure $C_{\mathscr{m}}(M)$ of a subset M of X consists of all points $x \in X$ such that every $G_{\mathscr{m}}$-set containing x meets M (cf. [25]). Clearly $C_{\mathscr{m}}$ is a hereditary Kuratowski operator. $\mathscr{D}(C_{\mathscr{m}})$ is the category of all spaces in which distinct points can be separated by disjoint $G_{\mathscr{m}}$-sets. Obviously **Haus** $\subset \mathscr{D}(C_{\mathscr{m}}) \subset \{T_1\text{-spaces}\}$. $C_{\mathscr{m}}$ coincides with $C_{\mathscr{D}(C_{\mathscr{m}})}$ in the subcategory $\mathscr{D}(C_{\mathscr{m}})$.

(5) Finally we observe that a regular closure operator C in **Top** coincides with the trivial closure on a space X if and only if the reflection of X in $\mathscr{D}(C)$ is one point. This is equivalent to $X \in \mathscr{C}(C)$, therefore $\mathscr{C}(C)$ is productive.

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